Center Manifold Theorem for Integral Equations (Global qualitative theory of ordinary differential equations and its applications)

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Center Manifold Theorem for Integral Equations

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1 Introduction

In this paper we are concerned with the integral equation (with infinite delay)

$$x(t)=\int_{-\infty}^{t}K(t-s)x(s)ds+f(x_t), \tag{E}$$

where K is a measurable $m\times m$ matrix valued function with complex components satisfying the condition $\int_0^\infty\|K(t)\|e^{\rho t}dt<\infty$ and ess $\sup\{\|K(t)\|e^{\rho t}:t\geq 0\}<\infty$, and f is a nonlinear term belonging to the space $C^1(X;\mathbb{C}^m)$, the set of all continuously (Fréchet) differentiable functions mapping X into $\mathbb{C}^m$, with the property that $f(0)=0$ and $Df(0)=0$; here, ρ is a positive constant which is fixed throughout the paper, and $X:=L_\rho^1(\mathbb{R}^-;\mathbb{C}^m)$, $\mathbb{R}^-:=(-\infty,0]$, is a Banach space (employed throughout the paper as the phase space for Eq. (E)) equipped with norm $\|\phi\|_X:=\int_{-\infty}^0|\phi(\theta)|e^{\rho\theta}d\theta$ $(\forall\phi\in X)$, and x_t is an element in X defined as $x_t(\theta)=x(t+\theta)$ for $\theta\in\mathbb{R}^-$. The linearized equation of Eq. (E) (around the equilibrium point 0) is given by

$$x(t)=\int_{-\infty}^{t}K(t-s)x(s)ds, \tag{1}$$

which possesses the characteristic matrix $\Delta(\lambda):=E_m-\int_0^\infty K(t)e^{-\lambda t}dt$ $(\mathrm{Re}\,\lambda>-\rho)$; here E_m is the $m\times m$ unit matrix. Recently, Diekmann and Gyllenberg [3] have treated Eq. (E), and established the principle of linearized stability for integral equations. In the paper, as a further development in the stability problem of Eq. (E), we treat the case that the equilibrium point zero is nonhyperbolic (that is, the set $\{\lambda\in\mathbb{C}:\det\Delta(\lambda)=0\ \&\ \mathrm{Re}\,\lambda=0\}$ is nonempty), and establish center manifold theorem for Eq. (E); and then we will investigate stability properties of the zero solution of Eq. (E) in the critical case.

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2 Several preparatory results for integral equations

In this section, following [6] we summarize several preliminary results necessary for our later arguments. Eq.(E) can be formulated as an abstract equation on the space X of the form

$$x(t) = L(x_t) + f(x_t),$$

where $L : X \to \mathbb{C}^m$ is a bounded linear operator defined by $L(\phi) := \int_{-\infty}^{0} K(-\theta)\phi(\theta)d\theta$ for $\phi \in X$. Let us consider Eq.(E) with the initial condition

$$x_\sigma = \phi, \quad \text{that is,} \quad x(\sigma + \theta) = \phi(\theta) \quad \text{for } \theta \in \mathbb{R}^-, \tag{2}$$

where $(\sigma, \phi) \in \mathbb{R} \times X$ is given arbitrarily. A function $x : (-\infty, a) \to \mathbb{C}^m$ is said to be a solution of the initial value problem (E)-(2) on the interval (σ, a) if x satisfies the following conditions: (i) $x_\sigma = \phi$, that is, $x(\sigma + \theta) = \phi(\theta)$ for $\theta \in \mathbb{R}^-$; (ii) $x \in L^1_{\mathrm{loc}}[\sigma, a)$, x is locally integrable on $[\sigma, a)$; (iii) $x(t) = L(x_t) + f(x_t)$ for $t \in (\sigma, a)$.

By virtue of [6, Proposition 1], the initial value problem (E)-(2) has a unique (local) solution which is denoted by $x(t; \sigma, \phi, f)$; in fact, $x(t; \sigma, \phi, f)$ is defined globally if, in particular, $f(\phi)$ is globally Lipschitz continuous in ϕ. Moreover we remark that if $x(t)$ is a solution of Eq.(E) on (σ, a), then x_t is an X-valued continuous function on $[\sigma, a)$. Now suppose that $\phi = \psi$ in X, that is, $\phi(\theta) = \psi(\theta)$ a.e. $\theta \in \mathbb{R}^-$. Then by the uniqueness of solutions of (E)-(2) it follows that $x(t; \sigma, \phi, f) = x(t; \sigma, \psi, f)$ for $t \in (\sigma, a)$, so that $x_t(\sigma, \phi, f) = x_t(\sigma, \psi, f)$ in X for $t \in [\sigma, a)$. In particular, given $\sigma \in \mathbb{R}$, $x_t(\sigma, \cdot, f)$ induces a transformation on X for each $t \in [\sigma, a)$ provided that $x(t; \sigma, \phi, f)$ is the solution of (E)-(2) on (σ, a).

For any $t \geq 0$ and $\phi \in X$, we define $T(t)\phi \in X$ by

$$\big[T(t)\phi\big](\theta) := x_t(\theta; 0, \phi, 0) = \begin{cases} x(t + \theta; 0, \phi, 0), & -t < \theta \leq 0, \\ \phi(t + \theta), & \theta \leq -t. \end{cases}$$

Then $T(t)$ defines a bounded linear operator on X. In fact, $\{T(t)\}_{t \geq 0}$ is a strongly continuous semigroup of bounded linear operators on X, called the solution semigroup for Eq.(1). Denote by A the generator of $\{T(t)\}_{t\geq 0}$, and let $\sigma(A)$ and $P_\sigma(A)$ be the spectrum and the point spectrum of the generator A, respectively. Between the spectrum of A and the characteristic roots of Eq. (1), the relation $\sigma(A) \cap \mathbb{C}_{-\rho} = P_\sigma(A) \cap \mathbb{C}_{-\rho} = \{\lambda \in \mathbb{C}_{-\rho} : \det \Delta(\lambda) = 0\} (=: \Sigma)$ holds, where $\mathbb{C}_{-\rho} := \{z \in \mathbb{C} : \mathrm{Re}\, z > -\rho\}$. Moreover, for ess (A), the essential spectrum of A, we have the estimate $\sup_{\lambda \in \mathrm{ess}\,(A)} \mathrm{Re}\, \lambda \leq -\rho$. Now set $\Sigma^u := \{\lambda \in \sigma(A) : \mathrm{Re}\, \lambda > 0\}$, $\Sigma^c := \{\lambda \in \sigma(A) : \mathrm{Re}\, \lambda = 0\}$, and $\Sigma^s := \sigma(A) \backslash (\Sigma^c \cup \Sigma^u)$. Then these observations, combined with the analyticity of $\det \Delta(\lambda)$ on the domain $\mathbb{C}_{-\rho}$, yield the following result ([6, Theorem 2]):

Proposition 1. *Let $\{T(t)\}_{t\geq 0}$ be the solution semigroup of Eq.(1). Then X is decomposed as a direct sum of closed subspaces E^u, E^c, and E^s*

$$X = E^u \oplus E^c \oplus E^s$$

with the following properties:

(i) $\dim (E^u \oplus E^c) < \infty$,

(ii) $T(t)E^u \subset E^u$, $T(t)E^c \subset E^c$, *and* $T(t)E^s \subset E^s$ *for* $t \in \mathbb{R}^+ := [0, \infty)$,

(iii) $\sigma(A|_{E^u}) = \Sigma^u$, $\sigma(A|_{E^c}) = \Sigma^c$ *and* $\sigma(A|_{E^s \cap \mathcal{D}(A)}) = \Sigma^s$,

(iv) $T^u(t) := T(t)|_{E^u}$ *and* $T^c(t) := T(t)|_{E^c}$ *are extendable for* $t \in \mathbb{R} := (-\infty, \infty)$ *as groups of bounded linear operators on* E^u *and* E^c, *respectively,*

(v) $T^s(t) := T(t)|_{E^s}$ *is a strongly continuous semigroup of bounded linear operators on* E^s, *and its generator is identical with* $A|_{E^s \cap \mathcal{D}(A)}$,

(vi) *there exist positive constants* α, ε *with* $\alpha > \varepsilon$ *and a constant* $C \geq 1$ *such that*

$$\begin{aligned} \|T^s(t)\|_{\mathcal{L}(X)} &\leq Ce^{-\alpha t}, \quad t \in \mathbb{R}^+, \\ \|T^u(t)\|_{\mathcal{L}(X)} &\leq Ce^{\alpha t}, \quad\ \ t \in \mathbb{R}^-, \\ \|T^c(t)\|_{\mathcal{L}(X)} &\leq Ce^{\varepsilon |t|}, \quad t \in \mathbb{R}. \end{aligned}$$

In (vi) we note that C is a constant depending only on α and ε, and that the value of $\varepsilon > 0$ can be taken arbitrarily small. Also, we will use the notations $E^{cu} = E^c \oplus E^u$, $E^{su} = E^s \oplus E^u$ etc, and denote by Π^s the projection from X onto E^s along E^{cu}, and similarly for Π^u, Π^{cu} etc.

We now introduce a continuous function $\Gamma^n : \mathbb{R}^- \to \mathbb{R}^+$ for each natural number n which is of compact support with *support* $\Gamma^n \subset [-1/n, 0]$ and satisfies $\int_{-\infty}^0 \Gamma^n(\theta)d\theta = 1$. Notice that $\Gamma^n\beta \in X$ for any $\beta \in \mathbb{C}^m$. Let us recall that $x(\cdot; \sigma, \varphi, p)$ is the (unique) solution of the integral equation

$$x(t) = \int_{-\infty}^t K(t-s)x(s)ds + p(t), \quad t > \sigma \tag{3}$$

through (σ, φ); here $\varphi \in X$. The following result ([6, Theorem 3]), which will often be referred to as *VCF* for short, gives a representation formula for $x_t(\sigma, \varphi, p)$ in the space X by using $T(t)$, φ and p.

Proposition 2. *Let* $p \in C([\sigma, \infty); \mathbb{C}^m)$. *Then*

$$x_t(\sigma, \varphi, p) = T(t-\sigma)\varphi + \lim_{n\to\infty} \int_\sigma^t T(t-s)(\Gamma^n p(s))ds, \quad \forall t \geq \sigma \tag{4}$$

in X.

Let us consider a subset $\bar{X}$ consisting of all elements $\phi \in X$ which are continuous on $[-\varepsilon_\phi, 0]$ for some $\varepsilon_\phi > 0$, and set

$$X_0 = \{\varphi \in X \mid \varphi = \phi \text{ a.e. on } \mathbb{R}^- \text{ for some } \phi \in \bar{X}\}.$$

For any $\varphi \in X_0$, we define *the value of φ at zero* by $\varphi[0] = \phi(0)$, where ϕ is an element belonging to $\bar{X}$ satisfying $\phi = \varphi$ a.e. on $\mathbb{R}^-$. We note that the value $\varphi[0]$ is well-defined; that is, it does not depend on the particular choice of ϕ since $\phi(0) = \psi(0)$ for any other $\psi \in \bar{X}$ such that $\phi = \psi$ a.e. on $\mathbb{R}^-$. It is clear that X_0 is a normed space equipped with norm

$$\|\varphi\|_{X_0} := \|\varphi\|_X + |\varphi[0]|, \quad \forall \varphi \in X_0.$$

We note that the solution $x(\cdot;\sigma,\psi,p)$ of Eq. (3) through $(\sigma,\psi) \in \mathbb{R} \times X$ satisfies the relation $x_t(\sigma,\psi,p) \in X_0$ with $(x_t(\sigma,\psi,p))[0] = x(t;\sigma,\psi,p)$ whenever $t > \sigma$.

The following lemma can be established by applying Proposition 2 and [6, Theorem 4]. We omit the proof.

Lemma 1. *Let $f_* \in C(X;\mathbb{C}^m)$, and consider the equation*

$$x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds + f_*(x_t). \tag{E_*}$$

Moreover, let $\psi \in E^c$, and η be a constant such that $\varepsilon < \eta < \alpha$. Then we have:

(i) *If $x(t)$ is a solution of Eq. (E_*) defined on $\mathbb{R}$ with the properties that $\Pi^c x_0 = \psi$, $\sup_{t\in\mathbb{R}} \|x_t\|_X\, e^{-\eta|t|} < \infty$ and $\sup_{t\in\mathbb{R}} |f_*(x_t)| < \infty$, then the X-valued function $u(t) := x_t$ satisfies*

$$\begin{aligned} u(t) = T^c(t)\psi &+ \lim_{n\to\infty} \int_0^t T^c(t-s)\Pi^c\Gamma^n f_*(u(s))ds \\ &- \lim_{n\to\infty} \int_t^{\infty} T^u(t-s)\Pi^u\Gamma^n f_*(u(s))ds + \lim_{n\to\infty} \int_{-\infty}^{t} T^s(t-s)\Pi^s\Gamma^n f_*(u(s))ds \end{aligned}$$

for $t \in \mathbb{R}$, and moreover u belongs to $C(\mathbb{R}; X_0)$.

(ii) *Conversely, if $y \in C(\mathbb{R}; X)$ with $\sup_{t\in\mathbb{R}} \|y(t)\|_X\, e^{-\eta|t|} < \infty$ and $\sup_{t\in\mathbb{R}} |f_*(y(t))| < \infty$ satisfies*

$$\begin{aligned} y(t) = T^c(t)\psi &+ \lim_{n\to\infty} \int_0^t T^c(t-s)\Pi^c\Gamma^n f_*(y(s))d\tau \\ &- \lim_{n\to\infty} \int_t^{\infty} T^u(t-s)\Pi^u\Gamma^n f_*(y(s))ds + \lim_{n\to\infty} \int_{-\infty}^{t} T^s(t-s)\Pi^s\Gamma^n f_*(y(s))ds \end{aligned}$$

for $t \in \mathbb{R}$, then y belongs to $C(\mathbb{R}; X_0)$ and the function $\xi(t)$ defined by

$$\xi(t) := \big(y(t)\big)[0], \quad t \in \mathbb{R}$$

is a solution of Eq.(E_*) on $\mathbb{R}$ satisfying $\Pi^c \xi_0 = \psi$, $\sup_{t\in\mathbb{R}} \|\xi_t\|_X\, e^{-\eta|t|} < \infty$ and $\xi_t = y(t)$ for $t \in \mathbb{R}$.

3 Center manifold and its exponential attractivity

In what follows we assume that $f \in C^1(X;\mathbb{C}^m)$ satisfies $f(0)=0$ and $Df(0)=0$. In this section we will establish the existence of local center manifolds of the equilibrium point 0 of Eq.(E) and study their properties. To do so, in parallel with Eq.(E), we will consider a modified equation of (E) of the form

$$x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds + f_\delta(x_t), \tag{E_δ}$$

where f_δ with $\delta > 0$ is a modification of the original nonlinear term f; more precisely let $\chi : \mathbb{R} \to [0,1]$ be a C^∞-function such that $\chi(t) = 1$ $(|t| \le 2)$ and $\chi(t) = 0$ $(|t| \ge 3)$, and define

$$f_\delta(\phi) := \chi\big(\|\Pi^{su}\phi\|_X/\delta\big)\chi\big(\|\Pi^{c}\phi\|_X/\delta\big)f(\phi), \quad \phi \in X.$$

The function $f_\delta : X \to \mathbb{C}^m$ is continuous on X, and is of class C^1 when restricted to the open set $S_\delta := \{\phi \in X : \|\Pi^{su}\phi\|_X < \delta\}$ since we may assume that $\|\Pi^c\phi\|_X$ is of class C^1 for $\phi \ne 0$ because of $\dim E^c < \infty$. Moreover, by the assumption $f(0) = Df(0) = 0$, there exist a $\delta_1 > 0$ and a nondecreasing continuous function $\zeta_* : (0,\delta_1] \to \mathbb{R}^+$ such that $\zeta_*(+0) = 0$,

$$\|f_\delta(\phi)\|_X \le \delta\zeta_*(\delta) \quad \text{and} \quad \|f_\delta(\phi) - f_\delta(\psi)\|_X \le \zeta_*(\delta)\|\phi - \psi\|_X \tag{5}$$

for $\phi, \psi \in X$ and $\delta \in (0,\delta_1]$. Indeed, we may put

$$\zeta_*(\delta) = \big(\sup_{\|\phi\|_X \le 3\delta} \|Df(\phi)\|_{\mathcal{L}(X;\mathbb{C}^m)}\big)\cdot\big(1 + 3\sup_{0\le t\le 3}|\chi'(t)|\big)$$

(cf. [2, Lemma 4.1]). Taking $\delta_1 > 0$ small, we may also assume that there exists a positive number $M_1(\delta_1) =: M_1$ such that

$$\|Df_\delta(\phi)\|_{\mathcal{L}(X;\mathbb{C}^m)} \le M_1, \quad \phi \in S_\delta \tag{6}$$

for any $\delta \in (0,\delta_1]$. Fix a positive number η such that

$$\varepsilon < \eta < \alpha,$$

where ε and α are the constants in Proposition 1.

For the existence of center manifold for Eq.(E_δ) and its exponential attractivity, we have the following:

Theorem 1. *There exist a positive number δ and a C^1-map $F_{*,\delta} : E^c \to E^{su}$ with $F_{*,\delta}(0) = 0$ such that the following properties hold:*

(i) $W^c_\delta :=$ graph $F_{*,\delta}$ *is tangent to E^c at zero,*

(ii) W^c_δ *is invariant for Eq.* (E_δ), *that is, if* $\xi \in W^c_\delta$, *then* $x_t(0,\xi,f) \in W^c_\delta$ *for* $t \in \mathbb{R}$.

(iii) *Assume moreover that* $\Sigma^u = \emptyset$. *Then there exists a positive constant* β_0 *with the property that if* x *is a solution of Eq.*(E_δ) *on an interval* $J = [t_0, t_1]$, *then the inequality*

$$\|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X \leq C\|\Pi^s x_{t_0} - F_{*,\delta}(\Pi^c x_{t_0})\|_X e^{-\beta_0(t-t_0)}, \quad t \in J$$

holds true. In particular, if x *is a solution on an interval* $[t_0, \infty)$, x_t *tends to* W^c_δ *exponentially as* $t \to \infty$.

As will be shown in Proposition 4 given later, the map $F_{*,\delta} : E^c \to E^{su}$ in Theorem 1 is globally Lipschitz continuous with the Lipschitz constant $L(\delta) = 4C^2C_1\zeta_*(\delta)/(\alpha - \eta)$. Noticing that $L(\delta) \to 0$ as $\delta \to 0$, one can assume that the number δ satisfies $\delta \in (0, \delta_1]$ together with $L(\delta) \leq 1$. Let us take a small $r \in (0, \delta)$ so that $\|F_{*,\delta}(\psi)\|_X < \delta$ for any $\psi \in B_{E^c}(r) := \{\phi \in E^c : \|\phi\|_X < r\}$. Such a choice of r is possible by the continuity of $F_{*,\delta}$. Set $F_* := F_{*,\delta}|_{B_{E^c}(r)}$ and consider an open neighborhood Ω_0 of 0 in X defined by

$$\Omega_0 := \{\phi \in X : \|\Pi^{su}\phi\|_X < \delta, \ \|\Pi^c\phi\|_X < r\}.$$

Observe that $f \equiv f_\delta$ on Ω_0. Then the following theorem which yields a local center manifold for Eq. (E) as the graph of F_* immediately follows from Theorem 1.

Theorem 2. *Assume that* $f \in C^1(X; \mathbb{C}^m)$ *with* $f(0) = Df(0) = 0$. *Then there exist positive numbers* r, δ, *and a* C^1*-map* $F_* : B_{E^c}(r) \to E^{su}$ *with* $F_*(0) = 0$, *together with an open neighborhood* Ω_0 *of* 0 *in* X, *such that the following properties hold:*

(i) $W^c_{\mathrm{loc}}(r,\delta) := \operatorname{graph} F_*$ *is tangent to* E^c *at zero,*

(ii) $W^c_{\mathrm{loc}}(r,\delta)$ *is locally invariant for Eq.* (E), *that is,*

(a) *for any* $\xi \in W^c_{\mathrm{loc}}(r,\delta)$ *there exists a* $t_\xi > 0$ *such that* $x_t(0,\xi,f) \in W^c_{\mathrm{loc}}(r,\delta)$ *for* $|t| \leq t_\xi$,

(b) *if* $\xi \in W^c_{\mathrm{loc}}(r,\delta)$ *and* $x_t(0,\xi,f) \in \Omega_0$ *for* $0 \leq t \leq T$, *then* $x_t(0,\xi,f) \in W^c_{\mathrm{loc}}(r,\delta)$ *for* $0 \leq t \leq T$.

(iii) *Assume moreover that* $\Sigma^u = \emptyset$. *Then there exists a positive constant* β_0 *with the property that if* x *is a solution of Eq.*(E) *on an interval* $J = [t_0, t_1]$ *satisfying* $x_t \in \Omega_0$ *on* J, *then the inequality*

$$\|\Pi^s x_t - F_*(\Pi^c x_t)\|_X \leq C\|\Pi^s x_{t_0} - F_*(\Pi^c x_{t_0})\|_X e^{-\beta_0(t-t_0)}, \quad t \in J$$

holds true. In particular, if the solution $x(t)$ *is defined on* $[t_0, \infty)$ *satisfying* $x_t \in \Omega_0$ *on* $[t_0, \infty)$, *then* x_t *tends to* $W^c_{\mathrm{loc}}(r,\delta)$ *exponentially as* $t \to \infty$.

In what follows, we will prove Theorem 1 by establishing several propositions. We now take a $\delta_1 > 0$ sufficiently small so that

$$\zeta_*(\delta_1)CC_1\left(\frac{1}{\eta-\varepsilon}+\frac{2}{\alpha+\eta}+\frac{2}{\alpha-\eta}\right)<\frac{1}{2} \tag{7}$$

holds, and let $\delta \in (0, \delta_1]$. Also, let us consider the Banach space Y_η defined by

$$Y_\eta := \{y \in C(\mathbb{R}; X) : \sup_{t\in\mathbb{R}} \|y(t)\|_X\, e^{-\eta|t|} < \infty\}$$

with norm $\|y\|_{Y_\eta} := \sup_{t\in\mathbb{R}} \|y(t)\|_X\, e^{-\eta|t|}$, $y \in Y_\eta$. For any $(\psi, y) \in E^c \times Y_\eta$, we set

$$\begin{aligned}\mathcal{F}_\delta(\psi, y)(t) := T^c(t)\psi &+ \lim_{n\to\infty}\int_0^t T^c(t-s)\Pi^c\Gamma^n f_\delta(y(s))ds\\ &- \lim_{n\to\infty}\int_t^\infty T^u(t-s)\Pi^u\Gamma^n f_\delta(y(s))ds\\ &+ \lim_{n\to\infty}\int_{-\infty}^t T^s(t-s)\Pi^s\Gamma^n f_\delta(y(s))ds\end{aligned} \tag{8}$$

for $t \in \mathbb{R}$. Notice that the right-hand side is well-defined and that $\mathcal{F}_\delta(\psi, y)$ is an X-valued function on $\mathbb{R}$ for each $(\psi, y) \in E^c \times Y_\eta$. It is straightforward to certify that $\mathcal{F}_\delta(\psi, y) \in Y_\eta$ by virtue of Proposition 1 and (5); in other words, $\mathcal{F}_\delta$ defines a map from $E^c \times Y_\eta$ to Y_η. In fact, for each $\psi \in E^c$, $\mathcal{F}_\delta(\psi, \cdot)$ is a contraction map from Y_η into itself with Lipschitz constant $1/2$, because of the inequality

$$\begin{aligned}\|\mathcal{F}_\delta(\psi, y_1) - \mathcal{F}_\delta(\psi, y_2)\|_{Y_\eta} &\le \sup_{t\in\mathbb{R}} e^{-\eta|t|}\left|\int_0^t CC_1\zeta_*(\delta)e^{-\varepsilon(t-s)}\|y_1-y_2\|_{Y_\eta}e^{\eta|s|}ds\right|\\ &+ \sup_{t\in\mathbb{R}} e^{-\eta|t|}\int_t^\infty CC_1\zeta_*(\delta)e^{\alpha(t-s)}\|y_1-y_2\|_{Y_\eta}e^{\eta|s|}ds\\ &+ \sup_{t\in\mathbb{R}} e^{-\eta|t|}\int_{-\infty}^t CC_1\zeta_*(\delta)e^{-\alpha(t-s)}\|y_1-y_2\|_{Y_\eta}e^{\eta|s|}ds\\ &\le \zeta_*(\delta_1)CC_1\left(\frac{1}{\eta-\varepsilon}+\frac{2}{\alpha+\eta}+\frac{2}{\alpha-\eta}\right)\|y_1-y_2\|_{Y_\eta}\\ &\le (1/2)\|y_1-y_2\|_{Y_\eta}\end{aligned}$$

for $y_1,\ y_2 \in Y_\eta$. Thus, the map $\mathcal{F}_\delta(\psi, \cdot)$ has a unique fixed point for each $\psi \in E^c$, say $\Lambda_{*,\delta}(\psi) \in Y_\eta$, i.e., we have

$$\begin{aligned}\Lambda_{*,\delta}(\psi)(t) = T^c(t)\psi &+ \lim_{n\to\infty}\int_0^t T^c(t-s)\Pi^c\Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s))ds\\ &- \lim_{n\to\infty}\int_t^\infty T^u(t-s)\Pi^u\Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s))ds\\ &+ \lim_{n\to\infty}\int_{-\infty}^t T^s(t-s)\Pi^s\Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s))ds\end{aligned} \tag{9}$$

for $t \in \mathbb{R}$, whenever $0 < \delta \le \delta_1$.

Proposition 3. $\Lambda_{*,\delta}(\psi)$ *satisfies the following:*

(i) $\|\Lambda_{*,\delta}(\psi_1) - \Lambda_{*,\delta}(\psi_2)\|_{Y_\eta} \le 2C\|\psi_1 - \psi_2\|_X$ *for* $\psi_1, \psi_2 \in E^c$.

(ii) $\Lambda_{*,\delta}(\psi)(t+\tau) = \Lambda_{*,\delta}\big(\Pi^c(\Lambda_{*,\delta}(\psi)(\tau))\big)(t)$ *holds for* $t, \tau \in \mathbb{R}$.

Proof. Since $\varepsilon < \eta$, (i) immediately follows from the estimate

$$\begin{aligned}
\|\Lambda_*(\psi_1) - \Lambda_*(\psi_2)\|_{Y_\eta} &= \|\mathcal{F}_\delta(\psi_1, \Lambda_{*,\delta}(\psi_1)) - \mathcal{F}_\delta(\psi_2, \Lambda_{*,\delta}(\psi_2))\|_{Y_\eta} \\
&\le \|\mathcal{F}_\delta(\psi_1, \Lambda_{*,\delta}(\psi_1)) - \mathcal{F}_\delta(\psi_1, \Lambda_{*,\delta}(\psi_2))\|_{Y_\eta} \\
&\quad + \|\mathcal{F}_\delta(\psi_1, \Lambda_{*,\delta}(\psi_2)) - \mathcal{F}_\delta(\psi_2, \Lambda_{*,\delta}(\psi_2))\|_{Y_\eta} \\
&\le (1/2)\|\Lambda_{*,\delta}(\psi_1) - \Lambda_{*,\delta}(\psi_2)\|_{Y_\eta} + \|T^c(\cdot)(\psi_1 - \psi_2)\|_{Y_\eta} \\
&\le (1/2)\|\Lambda_{*,\delta}(\psi_1) - \Lambda_{*,\delta}(\psi_2)\|_{Y_\eta} + \sup_{t\in\mathbb{R}} \big(Ce^{\varepsilon|t|}\|\psi_1 - \psi_2\|_X e^{-\eta|t|}\big).
\end{aligned}$$

Next, given $\tau \in \mathbb{R}$, let us consider the function $\tilde{\Lambda}(t)$ defined by $\tilde{\Lambda}(t) := \Lambda_{*,\delta}(\psi)(t+\tau)$, $t \in \mathbb{R}$. Obviously, $\tilde{\Lambda}(\cdot) \in Y_\eta$. Also, it is easy to check that $\tilde{\Lambda}(t) = \mathcal{F}_\delta\big(\Pi^c(\Lambda_{*,\delta}(\psi)(\tau)), \tilde{\Lambda}\big)(t)$ for all $t \in \mathbb{R}$; that is, $\tilde{\Lambda}$ is a fixed point of $\mathcal{F}_\delta\big(\Pi^c(\Lambda_{*,\delta}(\psi)(\tau)), \cdot\big)$. The uniqueness of the fixed points yields $\tilde{\Lambda} = \Lambda_{*,\delta}\big(\Pi^c(\Lambda_{*,\delta}(\psi)(\tau))\big)$, and hence

$$\Lambda_{*,\delta}(\psi)(t+\tau) = \tilde{\Lambda}(t) = \Lambda_{*,\delta}\big(\Pi^c(\Lambda_{*,\delta}(\psi)(\tau))\big)(t), \quad t \in \mathbb{R},$$

which shows (ii). $\square$

For $\delta \in (0, \delta_1]$ let $F_{*,\delta} : E^c \to E^{su}$ be the map defined by $F_{*,\delta}(\psi) := \Pi^{su} \circ \mathrm{ev}_0 \circ \Lambda_{*,\delta}(\psi)$ for $\psi \in E^c$, where ev_0 is the evaluation map: $\mathrm{ev}_0(y) := y(0)$ for $y \in C(\mathbb{R}; X)$. Then

$$\begin{aligned}
F_{*,\delta}(\psi) &= -\lim_{n\to\infty} \int_0^\infty T^u(-s)\Pi^u\Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s))ds \\
&\quad + \lim_{n\to\infty} \int_{-\infty}^0 T^s(-s)\Pi^s\Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s))ds, \quad \psi \in E^c; \qquad (10)
\end{aligned}$$

and in particular $\Lambda_{*,\delta}(\psi)(0) = \psi + F_{*,\delta}(\psi)$ for $\psi \in E^c$. Let us set

$$W^c_\delta := \operatorname{graph} F_{*,\delta} = \big\{\psi + F_{*,\delta}(\psi) : \psi \in E^c\big\}.$$

Proposition 4. *The map* $F_{*,\delta}$ *and its graph* W^c_δ *have the following properties:*

(i) $F_{*,\delta}$ *is (globally) Lipschitz continuous, i.e.,*

$$\|F_{*,\delta}(\psi_1) - F_{*,\delta}(\psi_2)\|_X \le L(\delta)\|\psi_1 - \psi_2\|_X, \quad \psi_1, \psi_2 \in E^c,$$

where $L(\delta) := 4C^2C_1\zeta_*(\delta)/(\alpha - \eta)$.

(ii) *Let $\hat{\phi} \in W^c_\delta$ and $\tau \in \mathbb{R}$. Then the solution of (E_δ) through $(\tau, \hat{\phi})$, $x(t; \tau, \hat{\phi}, f_\delta)$, exists on $\mathbb{R}$ and*

$$x_t(\tau, \hat{\phi}, f_\delta) = \Lambda_{*,\delta}(\hat{\psi})(t - \tau), \quad t \in \mathbb{R},$$

where $\hat{\psi} = \Pi^c \hat{\phi}$.

(iii) *Moreover for $\hat{\phi} \in W^c_\delta$ and $\tau \in \mathbb{R}$,*

$$\Pi^{su} x_t(\tau, \hat{\phi}, f_\delta) = F_{*,\delta}\big(\Pi^c x_t(\tau, \hat{\phi}, f_\delta)\big), \quad t \in \mathbb{R}.$$

In particular W^c_δ is invariant for (E_δ), that is, $x_t(\tau, \hat{\phi}, f_\delta) \in W^c_\delta$ for $t \in \mathbb{R}$, provided that $\hat{\phi} \in W^c_\delta$.

Proof. (i) By (10) and Proposition 3 (i), we get

$$\begin{aligned}\|F_{*,\delta}(\psi_1) - F_{*,\delta}(\psi_2)\|_X &\le \int_0^\infty CC_1 e^{-\alpha s} \zeta_*(\delta) \|\Lambda_{*,\delta}(\psi_1)(s) - \Lambda_{*,\delta}(\psi_2)(s)\|_X ds \\ &\quad + \int_{-\infty}^0 CC_1 e^{\alpha s} \zeta_*(\delta) \|\Lambda_{*,\delta}(\psi_1)(s) - \Lambda_{*,\delta}(\psi_2)(s)\|_X ds \\ &\le \frac{2CC_1\zeta_*(\delta)}{\alpha - \eta} \times 2C\|\psi_1 - \psi_2\|_X = L(\delta)\|\psi_1 - \psi_2\|_X,\end{aligned}$$

as required.
(ii) Applying Lemma 1 (i), we deduce that $\Lambda_{*,\delta}(\hat{\psi}) \in C(\mathbb{R}; X_0)$ and that the X-valued function $\xi(t) := \big(\Lambda_{*,\delta}(\hat{\psi})(t)\big)[0]$ $(t \in \mathbb{R})$ satisfies $\xi_t = \Lambda_{*,\delta}(\hat{\psi})(t)$ for $t \in \mathbb{R}$ and is a solution of (E_δ) on $\mathbb{R}$ with $\xi_0 = \Lambda_{*,\delta}(\hat{\psi})(0) = \hat{\psi} + F_{*,\delta}(\hat{\psi}) = \hat{\phi}$. Let $x(t) := \xi(t - \tau)$. Then $x(t)$ is a solution of (E_δ) on $\mathbb{R}$ with $x_\tau = \hat{\phi}$, so that $x(t) = x(t; \tau, \hat{\phi}, f_\delta)$ for $t \in \mathbb{R}$. Consequently,

$$x_t(\tau, \hat{\phi}, f_\delta) = \xi_{t-\tau} = \Lambda_{*,\delta}(\hat{\psi})(t - \tau), \quad t \in \mathbb{R}.$$

(iii) Notice from Proposition 3 (ii) that $\Lambda_{*,\delta}(\hat{\psi})(t - \tau) = \Lambda_{*,\delta}\big(\Pi^c(\Lambda_{*,\delta}(\hat{\psi})(t - \tau))\big)(0)$ for $\hat{\psi} := \Pi^c \hat{\phi}$, which, combined with (ii), yields that

$$\begin{aligned}\Pi^{su} x_t(\tau, \hat{\phi}, f_\delta) &= \Pi^{su}\big(\Lambda_{*,\delta}\big(\Pi^c(\Lambda_{*,\delta}(\hat{\psi})(t - \tau))\big)(0)\big) \\ &= \Pi^{su}\big(\Lambda_{*,\delta}\big(\Pi^c x_t(\tau, \hat{\phi}, f_\delta)\big)(0)\big) = F_{*,\delta}\big(\Pi^c x_t(\tau, \hat{\phi}, f_\delta)\big);\end{aligned}$$

which is the desired one. The latter part of (iii) is obvious. □

Now assume that $\Sigma^u = \emptyset$, i.e., $E^u = \{0\}$. Fix a $\delta \in (0, \delta_1]$ and let

$$K := CC_1\zeta_*(\delta), \quad \mu := K + \varepsilon.$$

Proposition 5. *Let $x(t)$ be a solution of (E_δ) on an interval $J := [t_0, t_1]$. Given $\tau \in J$, put $\hat{\phi} := \Pi^c x_\tau + F_{*,\delta}(\Pi^c x_\tau)$. Then the following inequalities hold:*

(i) *For* $t_0 \leq t \leq \tau$

$$\|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq K \int_t^\tau e^{\mu(s-t)} \|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X ds.$$

(ii) *Moreover for* $t_0 \leq t \leq \tau$

$$\|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq K \int_t^\tau e^{\mu'(s-t)} \|\xi(s)\|_X ds,$$

where $\mu' := \mu + KL(\delta)$ *and* $\xi(t) := \Pi^s x_t - F_{*,\delta}(\Pi^c x_t)$ *for* $t \in \mathbb{R}$.

Proof. By virtue of Proposition 4 (ii) and (iii), the solution $x(t; \tau, \hat{\phi}, f_\delta)$ exists on $\mathbb{R}$ and $\Pi^s x_t(\tau, \hat{\phi}, f_\delta) = F_{*,\delta}(\Pi^c x_t(\tau, \hat{\phi}, f_\delta))$ for $t \in \mathbb{R}$. Let $t_0 \leq t \leq \tau$. VCF gives

$$x_\tau(\tau, \hat{\phi}, f_\delta) = T(\tau - t)x_t(\tau, \hat{\phi}, f_\delta) + \lim_{n\to\infty} \int_t^\tau T(\tau - s)\Gamma^n f_\delta(x_s(\tau, \hat{\phi}, f_\delta))ds,$$

in particular

$$\Pi^c x_\tau(\tau, \hat{\phi}, f_\delta) = T^c(\tau - t)\Pi^c x_t(\tau, \hat{\phi}, f_\delta) + \lim_{n\to\infty} \int_t^\tau T^c(\tau - s)\Pi^c\Gamma^n f_\delta(x_s(\tau, \hat{\phi}, f_\delta))ds.$$

By the group property of $\{T^c(t)\}_{t\in\mathbb{R}}$, we get

$$\Pi^c x_t(\tau, \hat{\phi}, f_\delta) = T^c(t - \tau)\Pi^c x_\tau(\tau, \hat{\phi}, f_\delta) - \lim_{n\to\infty} \int_t^\tau T^c(t - s)\Pi^c\Gamma^n f_\delta(x_s(\tau, \hat{\phi}, f_\delta))ds. \quad (11)$$

Similarly for the solution $x(t)$

$$\Pi^c x_t = T^c(t - \tau)\Pi^c x_\tau - \lim_{n\to\infty} \int_t^\tau T^c(t - s)\Pi^c\Gamma^n f_\delta(x_s)ds.$$

Then, since $\Pi^c x_\tau(\tau, \hat{\phi}, f_\delta) = \Pi^c\hat{\phi} = \Pi^c x_\tau$, it follows that

$$\begin{aligned} e^{\varepsilon t}\|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq & \int_t^\tau K e^{\varepsilon s}\|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X ds \\ & + \int_t^\tau K e^{\varepsilon s}\|\Pi^c x_s - \Pi^c x_s(\tau, \hat{\phi}, f_\delta)\|_X ds \end{aligned}$$

for $t_0 \leq t \leq \tau$. Hence we get

$$e^{\varepsilon t}\|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq \int_t^\tau K e^{K(s-t)} e^{\varepsilon s}\|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X ds,$$

which implies (i).

Next we will verify (ii). By Proposition 4 (iii) and (i), we get $\|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X \leq \|\xi(s)\|_X + L(\delta)\|\Pi^c x_s - \Pi^c x_s(\tau, \hat{\phi}, f_\delta)\|_X$ for $s \in J$. Hence it follows from (i) that

$$\begin{aligned} e^{\mu t}\|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq & \int_t^\tau K e^{\mu s}\|\xi(s)\|_X ds \\ & + \int_t^\tau KL(\delta) e^{\mu s}\|\Pi^c x_s - \Pi^c x_s(\tau, \hat{\phi}, f_\delta)\|_X ds; \end{aligned}$$

then

$$e^{\mu t}\|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq \int_t^\tau K e^{KL(\delta)(s-t)} e^{\mu s}\|\xi(s)\|_X ds,$$

which implies (ii). □

Recall that

$$K := CC_1\zeta_*(\delta), \quad \mu := K + \varepsilon, \quad \mu' := \mu + KL(\delta) = K(1 + L(\delta)) + \varepsilon. \tag{12}$$

Proposition 6. *Assume that $\Sigma^u = \emptyset$ and $x(t)$ is a solution of (E_δ) on $J = [t_0, t_1]$. Define $\hat{x}_t \in W^c_\delta$ by $\hat{x}_t := \Pi^c x_t + F_{*,\delta}(\Pi^c x_t)$ for $t \in J$, and set $y(s;t) := \Pi^c x_s(t, \hat{x}_t, f_\delta)$ for $t \in J$ and $s \leq t$. Then the following inequality holds:*

$$\|y(s;t) - y(s;t_0)\|_X \leq K \int_{t_0}^t e^{\mu'(\theta - s)}\|\xi(\theta)\|_X d\theta, \quad s \leq t_0,$$

where $\xi(\theta) := \Pi^s x_\theta - F_{*,\delta}(\Pi^c x_\theta)$ for $\theta \in [t_0, t]$.

Proof. Suppose that $s \leq t_0$. By the same reasoning as (11)

$$\Pi^c x_s(t, \hat{x}_t, f_\delta) = T^c(s-t)\Pi^c \hat{x}_t - \lim_{n\to\infty}\int_s^t T^c(s-\sigma)\Pi^c\Gamma^n f_\delta(x_\sigma(t, \hat{x}_t, f_\delta))d\sigma. \tag{13}$$

Applying VCF to x_t and using $\Pi^c\hat{x}_\tau = \Pi^c x_\tau$ $(\tau \in J)$, we deduce that

$$\Pi^c\hat{x}_t = T^c(t - t_0)\Pi^c\hat{x}_{t_0} + \lim_{n\to\infty}\int_{t_0}^t T^c(t-\sigma)\Pi^c\Gamma^n f_\delta(x_\sigma)d\sigma,$$

and thus, (13) becomes

$$\begin{aligned}\Pi^c x_s(t, \hat{x}_t, f_\delta) = T^c(s - t_0)\Pi^c\hat{x}_{t_0} &+ \lim_{n\to\infty}\int_{t_0}^t T^c(s-\sigma)\Pi^c\Gamma^n f_\delta(x_\sigma)d\sigma \\ &- \lim_{n\to\infty}\int_s^t T^c(s-\sigma)\Pi^c\Gamma^n f_\delta(x_\sigma(t, \hat{x}_t, f_\delta))d\sigma, \quad t \in J.\end{aligned}$$

Therefore

$$\begin{aligned}\|y(s;t) - y(s;t_0)\|_X &= \|\Pi^c x_s(t, \hat{x}_t, f_\delta) - \Pi^c x_s(t_0, \hat{x}_{t_0}, f_\delta)\|_X \\ &= \Big\| \lim_{n\to\infty}\int_{t_0}^t T^c(s-\sigma)\Pi^c\Gamma^n f_\delta(x_\sigma)d\sigma \\ &\quad - \lim_{n\to\infty}\int_s^t T^c(s-\sigma)\Pi^c\Gamma^n f_\delta(x_\sigma(t, \hat{x}_t, f_\delta))d\sigma \\ &\quad + \lim_{n\to\infty}\int_s^{t_0} T^c(s-\sigma)\Pi^c\Gamma^n f_\delta(x_\sigma(t_0, \hat{x}_{t_0}, f_\delta))d\sigma \Big\|_X \\ &\leq \int_{t_0}^t CC_1 e^{\varepsilon|s-\sigma|}\zeta_*(\delta)\|x_\sigma - x_\sigma(t, \hat{x}_t, f_\delta)\|_X d\sigma \\ &\quad + \int_s^{t_0} CC_1 e^{\varepsilon|s-\sigma|}\zeta_*(\delta)\|x_\sigma(t_0, \hat{x}_{t_0}, f_\delta) - x_\sigma(t, \hat{x}_t, f_\delta)\|_X d\sigma. \end{aligned} \tag{14}$$

Observe that

$$\begin{aligned}\|x_\sigma - x_\sigma(t,\hat{x}_t,f_\delta)\|_X &\le \|\Pi^s x_\sigma - F_{*,\delta}(\Pi^c x_\sigma)\|_X + \|F_{*,\delta}(\Pi^c x_\sigma) - F_{*,\delta}(\Pi^c x_\sigma(t,\hat{x}_t,f_\delta))\|_X \\ &\quad + \|\Pi^c x_\sigma - \Pi^c x_\sigma(t,\hat{x}_t,f_\delta)\|_X \\ &\le \|\xi(\sigma)\|_X + (1+L(\delta))\|\Pi^c x_\sigma - \Pi^c x_\sigma(t,\hat{x}_t,f_\delta)\|_X, \end{aligned} \tag{15}$$

where we used Proposition 4 (i) and (iii). Note also that

$$\begin{aligned}\|x_\sigma(t_0,\hat{x}_{t_0},f_\delta) - x_\sigma(t,\hat{x}_t,f_\delta)\|_X &\le \|F_{*,\delta}(\Pi^c x_\sigma(t_0,\hat{x}_{t_0},f_\delta)) - F_{*,\delta}(\Pi^c x_\sigma(t,\hat{x}_t,f_\delta))\|_X \\ &\quad + \|\Pi^c x_\sigma(t_0,\hat{x}_{t_0},f_\delta) - \Pi^c x_\sigma(t,\hat{x}_t,f_\delta)\|_X \\ &\le (1+L(\delta))\|y(\sigma;t) - y(\sigma;t_0)\|_X. \end{aligned} \tag{16}$$

In view of (14), (15) and (16), combined with Proposition 5 (ii), we deduce

$$\begin{aligned}\|y(s;t) - y(s;t_0)\|_X &\le \int_{t_0}^{t} K e^{\varepsilon(\sigma-s)}\big(\|\xi(\sigma)\|_X + (1+L(\delta))\|\Pi^c x_\sigma - \Pi^c x_\sigma(t,\hat{x}_t,f_\delta)\|_X\big)d\sigma \\ &\quad + \int_{s}^{t_0} K e^{\varepsilon(\sigma-s)}(1+L(\delta))\|y(\sigma;t) - y(\sigma;t_0)\|_X d\sigma \\ &\le \int_{t_0}^{t} K e^{\varepsilon(\sigma-s)}\|\xi(\sigma)\|_X d\sigma \\ &\quad + \int_{t_0}^{t} K e^{\varepsilon(\sigma-s)}(1+L(\delta))\Big(K\int_{\sigma}^{t} e^{\mu'(\tau-\sigma)}\|\xi(\tau)\|_X d\tau\Big)d\sigma \\ &\quad + \int_{s}^{t_0} K e^{\varepsilon(\sigma-s)}(1+L(\delta))\|y(\sigma;t) - y(\sigma;t_0)\|_X d\sigma. \end{aligned} \tag{17}$$

Notice that the second term of the right-hand side becomes

$$K\int_{t_0}^{t}\big(e^{\varepsilon(t_0-s)+\mu'(\sigma-t_0)} - e^{\varepsilon(\sigma-s)}\big)\|\xi(\sigma)\|_X d\sigma$$

because of (12). So we see from (17) that for $s \le t_0$

$$\begin{aligned}e^{\varepsilon s}\|y(s;t) - y(s;t_0)\|_X &\le K\int_{t_0}^{t} e^{(\varepsilon-\mu')t_0+\mu'\sigma}\|\xi(\sigma)\|_X d\sigma \\ &\quad + K(1+L(\delta))\int_{s}^{t_0} e^{\varepsilon\sigma}\|y(\sigma;t) - y(\sigma;t_0)\|_X d\sigma. \end{aligned}$$

By Gronwall's inequality and (12)

$$\begin{aligned}e^{\varepsilon s}\|y(s;t) - y(s;t_0)\|_X &\le \Big(K\int_{t_0}^{t} e^{(\varepsilon-\mu')t_0+\mu'\sigma}\|\xi(\sigma)\|_X d\sigma\Big)e^{K(1+L(\delta))(t_0-s)} \\ &= Ke^{-(\mu'-\varepsilon)s}\int_{t_0}^{t} e^{\mu'\sigma}\|\xi(\sigma)\|_X d\sigma, \end{aligned}$$

which yields the the desired one. □

Proposition 7. *Assume that $\Sigma^u = \emptyset$, and let $\delta \in (0, \delta_1]$ be a sufficiently small number satisfying*

$$\max\left(\mu', \frac{K(\alpha-\varepsilon)}{\alpha-\mu'}\right) < \alpha. \tag{18}$$

If $x(t)$ is a solution of (E_δ) on $J = [t_0, t_1]$, then the function $\xi(t) := \Pi^s x_t - F_{*,\delta}(\Pi^c x_t)$ satisfies the inequality

$$\|\xi(t)\|_X \le C\|\xi(t_0)\|_X e^{-\beta_0(t-t_0)}, \quad t \in J,$$

where $\beta_0 := \alpha - K(\alpha-\varepsilon)/(\alpha-\mu') > 0$. If in particular $J = [t_0, \infty)$, $\operatorname{dist}(x_t, W_\delta^c)$ *tends to* 0 *exponentially as $t \to \infty$.*

Proof. By applying VCF, one can easily deduce the relation

$$\begin{aligned}\xi(t) - T^s(t-t_0)\xi(t_0) = \lim_{n\to\infty}\int_{t_0-t}^{0} & T^s(-s)\Pi^s\Gamma^n\big(f_\delta(x_{s+t}) - f_\delta(\Lambda_{*,\delta}(\Pi^c x_t)(s))\big)ds \\ + \lim_{n\to\infty}\int_{-\infty}^{t_0-t} & T^s(-s)\Pi^s\Gamma^n\big(f_\delta(\Lambda_{*,\delta}(\Pi^c x_{t_0})(t-t_0+s)) \\ & - f_\delta(\Lambda_{*,\delta}(\Pi^c x_t)(s))\big)ds, \qquad t \in J.\end{aligned}$$

If we set $\hat{x}_t := \Pi^c x_t + F_{*,\delta}(\Pi^c x_t)$ for $t \in J$, by Proposition 4 (ii)

$$\Lambda_{*,\delta}(\Pi^c x_t)(s) = x_s(0, \hat{x}_t, f_\delta) = x_{s+t}(t, \hat{x}_t, f_\delta)$$

and

$$\Lambda_{*,\delta}(\Pi^c x_{t_0})(t-t_0+s) = x_{t-t_0+s}(0, \hat{x}_{t_0}, f_\delta) = x_{s+t}(t_0, \hat{x}_{t_0}, f_\delta)$$

in particular for $s \in \mathbb{R}^-$. So

$$\begin{aligned}\xi(t) = T^s(t-t_0)\xi(t_0) + \lim_{n\to\infty}\int_{t_0-t}^{0} & T^s(-s)\Pi^s\Gamma^n\big(f_\delta(x_{s+t}) - f_\delta(x_{s+t}(t, \hat{x}_t, f_\delta))\big)ds \\ + \lim_{n\to\infty}\int_{-\infty}^{t_0-t} & T^s(-s)\Pi^s\Gamma^n\big(f_\delta(x_{s+t}(t_0, \hat{x}_{t_0}, f_\delta)) - f_\delta(x_{s+t}(t, \hat{x}_t, f_\delta))\big)ds,\end{aligned}$$

and thus

$$\begin{aligned}\|\xi(t)\|_X \le Ce^{-\alpha(t-t_0)}\|\xi(t_0)\|_X &+ \int_{t_0}^{t} Ke^{\alpha(\theta-t)}\|x_\theta - x_\theta(t, \hat{x}_t, f_\delta)\|_X d\theta \\ &+ \int_{-\infty}^{t_0} Ke^{\alpha(\theta-t)}\|x_\theta(t_0, \hat{x}_{t_0}, f_\delta) - x_\theta(t, \hat{x}_t, f_\delta)\|_X d\theta.\end{aligned}$$

Since $x_\theta(t, \hat{x}_t, f_\delta)$ $(t \in J,\ \theta \in \mathbb{R})$ can be written as

$$\begin{aligned}x_\theta(t, \hat{x}_t, f_\delta) &= \Pi^c x_\theta(t, \hat{x}_t, f_\delta) + \Pi^s x_\theta(t, \hat{x}_t, f_\delta) \\ &= \Pi^c x_\theta(t, \hat{x}_t, f_\delta) + F_{*,\delta}(\Pi^c x_\theta(t, \hat{x}_t, f_\delta))\end{aligned}$$

by Proposition 4 (iii), it follows from Proposition 4 (i) and Proposition 6 that for $\theta \leq t_0$

$$\begin{aligned}\|x_\theta(t_0, \hat{x}_{t_0}, f_\delta) - x_\theta(t, \hat{x}_t, f_\delta)\|_X &\leq \|\Pi^c x_\theta(t_0, \hat{x}_{t_0}, f_\delta) - \Pi^c x_\theta(t, \hat{x}_t, f_\delta)\|_X \\ &\quad + \|F_{*,\delta}(\Pi^c x_\theta(t_0, \hat{x}_{t_0}, f_\delta)) - F_{*,\delta}(\Pi^c x_\theta(t, \hat{x}_t, f_\delta))\|_X \\ &\leq (1 + L(\delta))\|y(\theta; t) - y(\theta; t_0)\|_X \\ &\leq (1 + L(\delta))K \int_{t_0}^t e^{\mu'(\tau-\theta)}\|\xi(\tau)\|_X d\tau,\end{aligned}$$

where $y(\theta; t)$ $(t \in J)$ is the one in Proposition 6. On the other hand, for $t_0 \leq \theta \leq t$

$$\begin{aligned}\|x_\theta - x_\theta(t, \hat{x}_t, f_\delta)\|_X &\leq \|\Pi^s x_\theta - F_{*,\delta}(\Pi^c x_\theta)\|_X + \|F_{*,\delta}(\Pi^c x_\theta) - F_{*,\delta}(\Pi^c x_\theta(t, \hat{x}_t, f_\delta))\|_X \\ &\quad + \|\Pi^c x_\theta - \Pi^c x_\theta(t, \hat{x}_t, f_\delta)\|_X \\ &\leq \|\xi(\theta)\|_X + (1 + L(\delta))\|\Pi^c x_\theta - \Pi^c x_\theta(t, \hat{x}_t, f_\delta)\|_X \\ &\leq \|\xi(\theta)\|_X + (1 + L(\delta))K \int_\theta^t e^{\mu'(\sigma-\theta)}\|\xi(\sigma)\|_X d\sigma,\end{aligned}$$

where we used Proposition 4 (i), (iii) and Proposition 5 (ii). Thus we have

$$\begin{aligned}\|\xi(t)\|_X &\leq Ce^{-\alpha(t-t_0)}\|\xi(t_0)\|_X \\ &\quad + \int_{t_0}^t Ke^{\alpha(\theta-t)}\left(\|\xi(\theta)\|_X + (1 + L(\delta))K \int_\theta^t e^{\mu'(\sigma-\theta)}\|\xi(\sigma)\|_X d\sigma\right)d\theta \\ &\quad + \int_{-\infty}^{t_0} Ke^{\alpha(\theta-t)}(1 + L(\delta))K\left(\int_{t_0}^t e^{\mu'(\tau-\theta)}\|\xi(\tau)\|_X d\tau\right)d\theta \\ &= Ce^{-\alpha(t-t_0)}\|\xi(t_0)\|_X + \left(K + \frac{K^2(1 + L(\delta))}{\alpha - \mu'}\right)\int_{t_0}^t e^{\alpha(\sigma-t)}\|\xi(\sigma)\|_X d\sigma,\end{aligned}$$

so that

$$e^{\alpha t}\|\xi(t)\|_X \leq Ce^{\alpha t_0}\|\xi(t_0)\|_X + \hat{K}\int_{t_0}^t e^{\alpha\sigma}\|\xi(\sigma)\|_X d\sigma,$$

where $\hat{K} := K + K^2(1 + L(\delta))/(\alpha - \mu')$. An application of Gronwall's inequality gives $e^{\alpha t}\|\xi(t)\|_X \leq Ce^{\alpha t_0}\|\xi(t_0)\|_X e^{\hat{K}(t-t_0)}$, and hence

$$\|\xi(t)\|_X \leq C\|\xi(t_0)\|_X e^{-(\alpha-\hat{K})(t-t_0)}, \quad t \in J,$$

which is the desired one because of $\hat{K} = K(\alpha - \varepsilon)/(\alpha - \mu') = \alpha - \beta_0$.

The latter part of the proposition is evident. This completes the proof. □

Proof of Theorem 1. The properties (ii) and (iii) of Theorem 1 are now immediate consequences of Propositions 4 and 7, respectively. We verify the property (i). Observe that Y_η is a subspace of $Y_{\eta'}$ if $\eta < \eta' < \alpha$, and denote the inclusion map by $\mathcal{J} : Y_\eta \to Y_{\eta'}$. By almost the same reasoning as in [8], we see that $\mathcal{J}\Lambda_{*,\delta}$ is C^1 smooth as a map from E^c to $Y_{\eta'}$; and hence $F_{*,\delta} = \Pi^{su} \circ \mathrm{ev}_0 \circ \mathcal{J}\Lambda_{*,\delta}$ is also C^1 smooth. Moreover, since

$$[[D(\mathcal{J}\Lambda_{*,\delta})(0)](t)]\psi = T^c(t)\psi, \quad \psi \in E^c,\ t \in \mathbb{R}$$

holds by virtue of $Df_\delta(0) = Df(0) = 0$, it follows that

$$DF_{*,\delta}(0)\psi = D(\Pi^{su} \circ \mathrm{ev}_0 \circ \mathcal{J}\Lambda_{*,\delta})(0)\psi = \Pi^{su}T^c(0)\psi = \Pi^{su}\psi = 0, \quad \psi \in E^c;$$

hence $DF_{*,\delta}(0) = 0$, which implies (i). □

4 Stability analysis of integral equations via central equations

Center manifolds play a crucial role in the stability analysis of systems around non-hyperbolic equilibria. Indeed, center manifolds for several kinds of equations allow us to reduce the stability analysis of an original system to that of its restriction to a center manifold; see e.g., [1, 4, 5, 9]. In this section, introducing an ordinary differential equation (called the "central equation" of Eq. (E)) which is expressed by using the explicit formula of the projection Π^c, we will establish the reduction principle for integral equations that the stability properties for the central equation imply those of Eq. (E) in the neighborhood of its zero solution.

Assume that $\Sigma^c \neq \emptyset$. Let $\{\phi_1, \ldots, \phi_{d_c}\}$ be a basis for E^c, where d_c is the dimension of E^c. Then based on the formal adjoint theory for Eq. (1) developed in [7], one can consider its dual basis as elements in the Banach space

$$X^\sharp := L^1_\rho(\mathbb{R}^+; (\mathbb{C}^*)^m) = \{\psi : \mathbb{R}^+ \to (\mathbb{C}^*)^m : \psi(\tau)e^{-\rho\tau} \text{ is integrable on } \mathbb{R}^+\}$$

with norm

$$\|\psi\|_{X^\sharp} := \int_0^\infty |\psi(\tau)|e^{-\rho\tau}d\tau, \quad \psi \in X^\sharp,$$

where $(\mathbb{C}^*)^m$ is the space of m-dimensional row vectors with complex components equipped with the norm which is compatible with the one in $\mathbb{C}^m$, that is, $|z^*z| \le |z^*|\,|z|$ for $z^* \in (\mathbb{C}^*)^m$ and $z \in \mathbb{C}^m$. To be more precise, if we set

$$\langle \psi, \phi \rangle := \int_{-\infty}^0 \left(\int_\theta^0 \psi(\xi - \theta)K(-\theta)\phi(\xi)d\xi \right) d\theta, \quad (\psi, \phi) \in X^\sharp \times X,$$

then this pairing defines a bounded bilinear form on $X^\sharp \times X$ with the property

$$|\langle \psi, \phi \rangle| \le \|K\|_{\infty,\rho}\|\psi\|_{X^\sharp}\|\phi\|_X, \quad (\psi, \phi) \in X^\sharp \times X;$$

here we recall that $\|K\|_{\infty,\rho} = \text{ess sup}\{\|K(t)\|e^{\rho t} : t \geq 0\}$. Then there exist $\{\psi_1, \ldots, \psi_{d_c}\}$, elements of $X^\sharp$, such that $\langle \psi_i, \phi_j \rangle = 1$ if $i = j$ and 0 otherwise, and $\langle \psi_i, \phi \rangle = 0$ for $\phi \in E^s$ and $i = 1, 2, \ldots, d_c$; we call $\{\psi_1, \ldots, \psi_{d_c}\}$ the dual basis of $\{\phi_1, \ldots, \phi_{d_c}\}$; see [7] for details. Denote by Φ_c and Ψ_c, $(\phi_1, \ldots, \phi_{d_c})$ and ${}^t(\psi_1, \ldots, \psi_{d_c})$, the transpose of $(\psi_1, \ldots, \psi_{d_c})$, respectively. Then, for any $\phi \in X$ the coordinate of its E^c-component with respest to the basis $\{\phi_1, \ldots, \phi_{d_c}\}$, or Φ_c for short, is given by $\langle \Psi_c, \phi \rangle := {}^t(\langle \psi_1, \phi \rangle, \ldots, \langle \psi_{d_c}, \phi \rangle) \in \mathbb{C}^{d_c}$, and therefore the projection Π^c is expressed, in terms of the basis Φ_c and its dual basis Ψ_c, by

$$\Pi^c \phi = \Phi_c \langle \Psi_c, \phi \rangle, \quad \phi \in X. \tag{19}$$

Since $\{T^c(t)\}_{t \geq 0}$ is a strongly continuous semigroup on the finite dimensional space E^c, there exists a $d_c \times d_c$ matrix G_c such that

$$T^c(t)\Phi_c = \Phi_c e^{tG_c}, \quad t \geq 0, \tag{20}$$

and $\sigma(G_c)$, the spectrum of G_c, is identical with Σ^c. The E^c-components of solutions of Eq.(E_δ) can be described by a certain ordinary differential equation in $\mathbb{C}^{d_c}$. More precisely, let $x(t)$ be a solution of Eq.(E_δ) through (σ, ϕ), that is, $x(t) = x(t; \sigma, \phi, f)$. If we denote by $z_c(t)$ the component of $\Pi^c x_t$ with respect to the basis Φ_c, that is, $\Phi_c z_c(t) := \Pi^c x_t$, or $z_c(t) := \langle \Psi_c, x_t \rangle$, then by virtue of [6, Theorem 7] $z_c(t)$ satisfies the ordinary differential equation

$$\dot{z}_c(t) = G_c z_c(t) + H_c f_\delta(\Phi_c z_c(t) + \Pi^{su} x_t), \tag{21}$$

where H_c is the $d_c \times m$ matrix such that $H_c x := \lim_{n \to \infty} \langle \Psi_c, \Gamma^n x \rangle$ for $x \in \mathbb{C}^m$.

In connection with Eq. (21), let us consider the ordinary differential equations on $\mathbb{C}^{d_c}$

$$\dot{z}(t) = G_c z(t) + H_c f_\delta(\Phi_c z(t) + F_{*,\delta}(\Phi_c z(t))) \tag{CE_δ}$$

and

$$\dot{z}(t) = G_c z(t) + H_c f(\Phi_c z(t) + F_*(\Phi_c z(t))). \tag{CE}$$

We call Eq. (CE) (resp. Eq. (CE_δ)) the central equation of (E) (resp. (E_δ)). Applying Proposition 4 (iii), one can easily derive the following result on relationships among solutions of Eq.(E_δ) (resp. Eq.(E)) and (CE_δ) (resp. (CE)).

Proposition 8. *The following statements hold true:*

(i) *Let x be a solution of Eq.(E_δ) on an interval J such that $x_t \in W_\delta^c$ $(t \in J)$. Then the function $z_c(t) := \langle \Psi_c, x_t \rangle$ satisfies the equation (CE_δ) on J.*
Conversely, if $z(t)$ satisfies the equation (CE_δ) on an interval J, then there exists a unique solution x of Eq.(E_δ) on J such that $x_t \in W_\delta^c$ and $\Pi^c x_t = \Phi_c z(t)$ on J.

(ii) *Let x be a solution of Eq.(E) on an interval J such that $x_t \in W^c_{\mathrm{loc}}(r,\delta)$ $(t \in J)$. Then the function $z_c(t) := \langle \Psi_c, x_t \rangle$ satisfies the equation (CE) on J, together with the inequality $\sup_{t\in J} \|\Phi_c z_c(t)\|_X \le r$.*
Conversely, if $z(t)$ satisfies the equation (CE) on an interval J together with the inequality $\sup_{t\in J} \|\Phi_c z(t)\|_X \le r$, then there exists a unique solution x of Eq.(E) on J such that $x_t \in W^c_{\mathrm{loc}}(r,\delta)$ and $\Pi^c x_t = \Phi_c z(t)$ on J.

Since $f(0) = f_\delta(0) = 0$, both equations (CE) and (CE_δ) (as well as (E) and (E_δ)) possess the zero solution. Notice that the zero solution of (CE) (resp. (E)) is uniformly asymptotically stable if and only if the zero solution of (CE_δ) (resp. (E_δ)) is uniformly asymptotically stable. Likewise, the zero solution of (CE) (resp. (E)) is unstable if and only if the zero solution of (CE_δ) (resp. (E_δ)) is unstable. Here, for the definition of several stability properties utilized in this paper, we refer readers to the books [10, 5].

Now suppose that $\Sigma^u = \emptyset$. Then the dynamics near the zero solution of (E) is determined by the dynamics near $z_c = 0$ of (CE) in the following sense.

Theorem 3. *Assume that $\Sigma^u = \emptyset$. If the zero solution of (CE) is uniformly asymptotically stable (resp. unstable), then the zero solution of (E) is also uniformly asymptotically stable (resp. unstable).*

Proof. By the fact stated in the preceding paragraph of the theorem, it is sufficient to establish that the uniform asymptotic stability (resp. instability) of the zero solution of (CE_δ) implies the uniform asymptotic stability (resp. instability) of the zero solution of (E_δ).

If the zero solution of (CE_δ) is unstable, the instability of the zero solution of (E_δ) immediately follows from the invariance of W^c_δ (Proposition 4 (iii)). In what follows, under the assumption that the the zero solution of (CE_δ) is uniformly asymptotically stable, we will establish the uniform asymptotic stability of the zero solution of (E_δ). By virtue of [5, Theorem 4.2.1], there exist positive constants a, $\bar{K}$ and a Liapunov function V defined on $S_a := \{y \in \mathbb{C}^{d_c} : |y| \le a\}$ satisfying the following properties:

(i) There exists a $b \in C(\mathbb{R}^+;\mathbb{R}^+)$ which is strictly increasing with $b(0) = 0$ and

$$b(|y|) \le V(y) \le |y| \quad \text{for} \quad y \in S_a.$$

(ii) $|V(y) - V(z)| \le \bar{K}|y - z|$ for $y, z \in S_a$.

(iii) $\dot{V}(z) \le -V(z)$ for $z \in S_a$, where $\dot{V}(z) := \limsup_{h\to+0}(1/h)\{V(y(h)) - V(z)\}$, and $y(h)$ is the solution of (CE_δ) with $y(0) = z$.

Choose a positive number τ_0 such that

$$e^{-\tau_0} \le \frac{1}{2} \quad \text{and} \quad Ce^{-\beta_0 \tau_0} \le \frac{1}{4}, \tag{22}$$

where β_0 is the one in Proposition 7, and we may assume that $\beta_0 > \mu'$, taking δ so small if necessary. Put $K_\infty := \|K\|_{\infty,\rho}$ and take a positive number P in such a way that

$$P > \max\Big(1, \frac{4C}{\beta_0 - \mu'}\bar{K} K K_\infty \|\Psi_c\|\Big), \tag{23}$$

and set $a_0 := ae^{-\eta\tau_0}/(4CK_\infty\|\Psi_c\|)$, where $\|\Psi_c\| := \big(\sum_{j=1}^{d_c} \|\psi_j\|^2_{X^\sharp}\big)^{1/2}$. Let Ω be a neighborhood of 0 in X such that

$$\langle \Psi_c, \phi\rangle \in S_a, \quad \|\Pi^c\phi\|_X \le a_0, \quad \text{and} \quad Q \le b(a)$$

for $\phi \in \Omega$, where

$$Q := V(\langle \Psi_c, \phi\rangle) + \Big(PC + \frac{\bar{K}K_\infty\|\Psi_c\|KC}{\beta_0 - \mu'}\Big)(\|\Pi^s\phi\|_X + \|F_{*,\delta}(\Pi^c\phi)\|_X),$$

and consider the function $W(\phi)$ on Ω defined by

$$W(\phi) := V(\langle \Psi_c, \phi\rangle) + P\|\Pi^s\phi - F_{*,\delta}(\Pi^c\phi)\|_X, \quad \phi \in \Omega.$$

W is continuous in Ω with $W(0) = 0$ and is positive in $\Omega\backslash\{0\}$ because of (i) and (ii).

We will first certify the following claim.

Claim 1. *There exists a positive number c_0 such that, for any $t_0 \in \mathbb{R}^+$ and $\phi \in X$ with $W(\phi) \le c_0$, the solution $x(t; t_0, \phi, f_\delta)$ exists on $[t_0, t_0 + \tau_0]$ and satisfies $x_t(t_0, \phi, f_\delta) \in \Omega$ for $t \in [t_0, t_0 + \tau_0]$; in particular, $\|\Pi^c x_t(t_0, \phi, f_\delta)\|_X \le a_0$ in this interval.*

Indeed, suppose that $x_t(t_0, \phi, f_\delta)$ is defined on the interval $[t_0, t_0 + t_*)$ with $t_* \le \tau_0$. Applying VCF, we get

$$\|x_t(t_0, \phi, f_\delta)\|_X \le M\|\phi\|_X + \int_{t_0}^t M\zeta_*(\delta)\|x_s(t_0, \phi, f_\delta)\|_X ds$$

for $t \in [t_0, t_0 + t_*)$, where $M := \sup_{0\le t\le\tau_0} \|T(t)\|_{\mathcal{L}(X)}$. Then Gronwall's inequality yields that $\|x_t(t_0, \phi, f_\delta)\|_X \le M\|\phi\|_X e^{M\zeta_*(\delta)(t-t_0)} \le M\|\phi\|_X e^{M\zeta_*(\delta)\tau_0}$ for $t \in [t_0, t_0 + t_*)$; which means that $x_t(t_0, \phi, f_\delta)$ can be defined on the interval $[t_0, t_0+t_*]$ and therefore on $[t_0, t_0+\tau_0]$ (cf. [6, Corollary 1]). Thus it turns out that if $\|\phi\|_X$ is small enough, $x_t(t_0, \phi, f_\delta)$ exists on $[t_0, t_0+\tau_0]$ and moreover belongs to Ω in this interval. The claim readily follows from the fact that $\inf\{W(\phi) : \phi \in \Omega, \|\phi\|_X \ge r\} > 0$ for small $r > 0$, together with the property of Ω.

Now given $t_0 \in \mathbb{R}^+$ and $\phi \in X$ with $W(\phi) \le c_0$, let us consider the solution $x(t) := x(t; t_0, \phi, f_\delta)$. By Proposition 3 (i)

$$\|\Lambda_{*,\delta}(\Pi^c x_t)(s)\|_X \le \|\Lambda_{*,\delta}(\Pi^c x_t)\|_{Y_\eta} e^{\eta|s|} \le e^{\eta|s|} 2C\|\Pi^c x_t\|_X, \quad s \in \mathbb{R};$$

hence taking account of $\Lambda_{*,\delta}(\Pi^c x_t)(s) = x_{t+s}(t, \hat{x}_t, f_\delta)$ for $s \in \mathbb{R}$ (Proposition 4 (ii)), we get $\|x_{t+s}(t, \hat{x}_t, f_\delta)\|_X \le e^{\eta\tau_0} 2C\|\Pi^c x_t\|_X$ for $s \in [-\tau_0, 0]$, where $\hat{x}_t := \Pi^c x_t + F_{*,\delta}(\Pi^c x_t)$. Set $y^\circ(t+s; t) := \langle \Psi_c, x_{t+s}(t, \hat{x}_t, f_\delta)\rangle$. Then $|y^\circ(t+s;t)| \le K_\infty \|\Psi_c\| \|x_{t+s}(t, \hat{x}_t, f_\delta)\|_X \le 2CK_\infty\|\Psi_c\| e^{\eta\tau_0}\|\Pi^c x_t\|_X \le 2CK_\infty \|\Psi_c\| e^{\eta\tau_0} a_0 = a/2$ for $s \in [-\tau_0, 0]$; hence $y^\circ(s;t) \in S_{a/2}$ and thus $V(y^\circ(s;t))$ is well-defined for $s \in [t_0, t]$ with $t \in [t_0, t_0 + \tau_0]$.

We next confirm:

Claim 2. $\sup\{W(x_t) : t \in [t_0, t_0+\tau_0]\} \le Q$ *and* $W(x_{t_0+\tau_0}(t_0, \phi, f_\delta)) \le c_0/2$.

Indeed, fix a $t \in [t_0, t_0 + \tau_0]$ and set $z(s) := y^\circ(s;t)$ for $s \in [t_0, t]$. Since $y^\circ(s;t) = \langle \Psi_c, x_s(t, \hat{x}_t, f_\delta)\rangle = \langle \Psi_c, \Pi^c x_s(t, \hat{x}_t, f_\delta)\rangle$ for $s \in [t_0, t]$, $z(s)$ is a solution of (CE_δ) on $[t_0, t]$ with $z(t) = y^\circ(t;t) = \langle \Psi_c, \Pi^c x_t\rangle$. By the property (i), we see that $\dot{V}(z(s)) \le -V(z(s))$ for $s \in [t_0, t]$, which implies that $(d/ds)\big(e^{s-t}V(z(s))\big) = e^{s-t}\big(V(z(s)) + \dot{V}(z(s))\big) \le 0$, so that

$$V(\langle \Psi_c, \Pi^c x_t\rangle) - e^{t_0 - t}V(y^\circ(t_0;t)) = V(z(t)) - e^{t_0-t}V(z(t_0)) \le \int_{t_0}^t \frac{d}{ds}\big(e^{s-t}V(z(s))\big)ds \le 0;$$

consequently,

$$\begin{aligned}
V(\langle \Psi_c, \Pi^c x_t\rangle) &\le e^{t_0-t}V(\langle \Psi_c, \Pi^c x_{t_0}\rangle) + e^{t_0-t}\big(V(y^\circ(t_0;t)) - V(\langle \Psi_c, \Pi^c x_{t_0}\rangle)\big) \\
&\le e^{t_0-t}V(\langle \Psi_c, \Pi^c x_{t_0}\rangle) + e^{t_0-t}\bar{K}\big|y^\circ(t_0;t) - \langle \Psi_c, \Pi^c x_{t_0}\rangle\big| \\
&\le e^{t_0-t}V(\langle \Psi_c, \Pi^c \phi\rangle) + e^{t_0-t}\bar{K}K_\infty\|\Psi_c\|\|\Pi^c x_{t_0}(t, \hat{x}_t, f_\delta) - \Pi^c x_{t_0}\|_X \\
&\le e^{t_0-t}V(\langle \Psi_c, \Pi^c \phi\rangle) + e^{t_0-t}\bar{K}K_\infty\|\Psi_c\|K\int_{t_0}^t e^{\mu'(\theta-t_0)}\|\xi(\theta)\|_X d\theta,
\end{aligned}$$

where the last inequality is due to Proposition 5 (ii). Therefore, applying Proposition 7,

$$\begin{aligned}
W(x_t) &= V(\langle \Psi_c, \Pi^c x_t\rangle) + P\|\xi(t)\|_X \\
&\le e^{t_0-t}V(\langle \Psi_c, \Pi^c \phi\rangle) + e^{t_0-t}\bar{K}K_\infty\|\Psi_c\|K\int_{t_0}^t e^{\mu'(\theta-t_0)}\big(C\|\xi(t_0)\|_X e^{-\beta_0(\theta-t_0)}\big)d\theta \\
&\quad + PC\|\xi(t_0)\|_X e^{-\beta_0(t-t_0)} \\
&\le e^{t_0-t}V(\langle \Psi_c, \Pi^c \phi\rangle) + C\|\xi(t_0)\|_X\left(\frac{\bar{K}K_\infty K\|\Psi_c\|}{\beta_0 - \mu'}e^{t_0-t} + Pe^{-\beta_0(t-t_0)}\right).
\end{aligned} \tag{24}$$

In particular,

$$\begin{aligned}
W(x_{t_0+\tau_0}) &\le e^{-\tau_0}V(\langle \Psi_c, \Pi^c \phi\rangle) + C\|\xi(t_0)\|_X\left(\frac{\bar{K}K_\infty K\|\Psi_c\|}{\beta_0 - \mu'}e^{-\tau_0} + Pe^{-\beta_0\tau_0}\right) \\
&\le (1/2)V(\langle \Psi_c, \Pi^c \phi\rangle) + (1/2)P\|\xi(t_0)\|_X \\
&= (1/2)W(x_{t_0}) = (1/2)W(\phi) \le (1/2)c_0.
\end{aligned}$$

Since $\|\xi(t_0)\|_X \le \|\Pi^s\phi\|_X + \|F_{*,\delta}(\Pi^c\phi)\|_X$, (24) implies also

$$\sup\{W(x_t) : t \in [t_0, t_0+\tau_0]\} \le V(\langle \Psi_c, \Pi^c\phi\rangle) + C\|\xi(t_0)\|_X \left(\frac{\bar{K}K_\infty K\|\Psi_c\|}{\beta_0 - \mu'} + P\right) \le Q,$$

as required.

By Claim 2, combined with Claim 1, $x(t) = x(t; t_0, \phi, f_\delta)$ is defined on $[t_0, t_0 + 2\tau_0]$, and $y^\circ(s;t) \in S_{a/2}$ still holds for $s \in [t_0, t]$ with $t \in [t_0, t_0 + 2\tau_0]$. More generally, one can deduce that $x(t) = x(t; t_0, \phi, f_\delta)$ is defined on $[t_0, t_0 + n\tau_0]$, and $y^\circ(s;t) \in S_{a/2}$ holds for $s \in [t_0, t]$ with $t \in [t_0, t_0 + n\tau_0]$ for any $n \in \mathbb{N}$, together with the relations

$$\sup\{W(x_t) : t \in [t_0 + (n-1)\tau_0, t_0 + n\tau_0]\} \le \frac{Q}{2^{n-1}} \quad \text{and} \quad W(x_{t_0+n\tau_0}) \le \frac{c_0}{2^n}$$

for $n \in \mathbb{N}$. This means that $x(t) = x(t; t_0, \phi, f_\delta)$ is actually defined on $[t_0, \infty)$ and that

$$V(\langle \Psi_c, x_t(t_0, \phi, f_\delta)\rangle) + P\|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X \le Q\, 2^{-(t-t_0)/\tau_0}, \quad t \in [t_0, \infty).$$

In view of (i) and $P > 1$, it follows that $b(|\langle \Psi_c, x_t(t_0, \phi, f_\delta)\rangle|) \le Q\, 2^{-(t-t_0)/\tau_0} \le b(a)$ and $\|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X \le Q\, 2^{-(t-t_0)/\tau_0}$. Since $\|\Pi^c x_t(t_0, \phi, f_\delta)\|_X = \|\Phi_c \langle \Psi_c, x_t(t_0, \phi, f_\delta)\rangle\|_X \le \|\Phi_c\|\, b^{-1}(Q\, 2^{-(t-t_0)/\tau_0})$ with $\|\Phi_c\| := \left(\sum_{j=1}^{d_c} \|\phi_j\|_X^2\right)^{1/2}$ and $\|\Pi^s x_t(t_0, \phi, f_\delta)\|_X \le \|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X + \|F_{*,\delta}(\Pi^c x_t)\|_X \le Q\, 2^{-(t-t_0)/\tau_0} + L(\delta)\|\Pi^c x_t\|_X$, we obtain that for any $\phi \in \Omega$ and $t \in [t_0, \infty)$

$$\begin{aligned} \|x_t(t_0, \phi, f_\delta)\|_X &\le \|\Pi^c x_t(t_0, \phi, f_\delta)\|_X + \|\Pi^s x_t(t_0, \phi, f_\delta)\|_X \\ &\le Q\, 2^{-(t-t_0)/\tau_0} + (1 + L(\delta))\|\Phi_c\|\, b^{-1}\big(Q\, 2^{-(t-t_0)/\tau_0}\big), \end{aligned}$$

which shows that the zero solution of (E_δ) is uniformly asymptotically stable. □

Before concluding this section, we will provide an example to illustrate how our Theorem 3 is available for stability analysis of some concrete equations. Let us consider nonlinear (scalar) integral equation

$$x(t) = \int_{-\infty}^{t} P(t-s)x(s)ds + f(x_t), \tag{25}$$

where P is a nonnegative continuous function on $\mathbb{R}^+$ satisfying $\int_0^\infty P(t)dt = 1$ together with the condition $\|P\|_{1,\rho} := \int_0^\infty P(t)e^{\rho t}dt < \infty$ and $\|P\|_{\infty,\rho} := \operatorname{ess\,sup}\{P(t)e^{\rho t} : t \ge 0\} < \infty$ for some positive constant ρ, and $f \in C^1(X; \mathbb{C})$, $X := L^1_\rho(\mathbb{R}^-; \mathbb{C})$, satisfies $f(0) = 0$ and $Df(0) = 0$. Eq. (25) is written as Eq. (E) with $m = 1$ and $K \equiv P$. The characteristic operator $\Delta(\lambda)$ of Eq. (25) is given by $\Delta(\lambda) = 1 - \int_0^\infty P(t)e^{-\lambda t}dt$. We thus get $\Sigma^u = \emptyset$ and $\Sigma^c = \{0\}$. Indeed, in this case, 0 is a simple root of the equation $\Delta(\lambda) = 0$, and E^c is 1-dimensional space with a basis $\{\phi_1\}$, $\phi_1 \equiv 1$, together with $\{\psi_1\}$, $\psi_1 \equiv 1/r$ (here

$r := \int_0^\infty \tau P(\tau)d\tau$), as the dual basis of $\{\phi_1\}$; see [7] for details. The projection Π^c is given by the formula $\Pi^c\phi = \Phi_c\langle\Psi_c, \phi\rangle$, $\forall\phi \in X$, and hence

$$\begin{aligned}\Pi^c\phi = \phi_1\langle\psi_1,\phi\rangle &= \phi_1\left(\int_{-\infty}^0\int_\theta^0 \psi_1(\xi-\theta)P(-\theta)\phi(\xi)d\xi d\theta\right)\\ &= \Phi_c\left(\frac{1}{r}\int_{-\infty}^0 P(-\theta)\left(\int_\theta^0 \phi(\xi)d\xi\right)d\theta\right).\end{aligned}$$

Thus, for a solution $x(t)$ of Eq. (25), the component $z_c(t)$ of $\Pi^c x_t$ with respect to Φ_c is given by

$$z_c(t) = \frac{1}{r}\int_{-\infty}^t \hat{P}(t-s)x(s)ds$$

with $\hat{P}(t) := \int_t^\infty P(\tau)d\tau$, because of

$$\begin{aligned}rz_c(t) &= \int_{-\infty}^0 P(-\theta)\left(\int_\theta^0 x(t+\xi)d\xi\right)d\theta = \int_{-\infty}^0 P(-\theta)\left(\int_{t+\theta}^t x(s)ds\right)d\theta\\ &= \int_{-\infty}^t P(t-\tau)\left(\int_\tau^t x(s)ds\right)d\tau = \int_{-\infty}^t\left(\int_{t-s}^\infty P(w)dw\right)x(s)ds.\end{aligned}$$

Observe that $z_c(t)$ satisfies the ordinary equation

$$r\dot{z}_c(t) = \hat{P}(0)x(t) + \int_{-\infty}^t(-P(t-s))x(s)ds = x(t) - \int_{-\infty}^t P(t-s)x(s)ds,$$

that is, $r\dot{z}_c(t) = f(x_t) = f(\Phi_c z_c(t) + \Pi^s x_t)$. In particular, if x is a solution of Eq. (25) satisfying $x_t \in W^c_{\mathrm{loc}}(r,\delta)$ on an interval J, then $\Pi^s x_t = F_*(\Phi_c z_c(t))$ on J; hence we get

$$\dot{z}_c(t) = (1/r)f(\Phi_c z_c(t) + F_*(\Phi_c z_c(t)))$$

on J. This observation leads to that $G_c = 0$ and $H_c = 1/r$ in the central equation (CE); in fact, by noticing that $\Sigma^c = \{0\}$ and $H_c x = \lim_{n\to+\infty}\langle\psi_1, \Gamma^n x\rangle = (1/r)x$, $\forall x \in \mathbb{C}$, one can also certify this fact. Consequently, the central equation of Eq. (25) is identical with the scalar equation $\dot{z} = H(z)$; here

$$H(w) := (1/r)f(\Phi_c w + F_*(\Phi_c w)), \quad (w \in \mathbb{C} \text{ and } |w| \text{ is small}).$$

In what follows, we will determine the function H for some special functions f.

Let us assume that f is of the form

$$f(\phi) = \varepsilon\left(\int_{-\infty}^0 Q(-\theta)\phi(\theta)d\theta\right)^m + g(\phi), \quad \forall\phi \in X, \tag{26}$$

where m is a natural number such that $m \geq 2$, ε is a nonzero real number, Q is a function satisfying $\|Q\|_{1,\rho} < \infty$ and $\|Q\|_{\infty,\rho} < \infty$ and $c_0 := \int_{-\infty}^0 Q(-\theta)d\theta > 0$, and

$g \in C^1(X;\mathbb{C})$ satisfies $|g(\phi)| = o(\|\phi\|_X^m)$ as $\|\phi\|_X \to 0$ (here, o means Landau's notation "small oh"). One can easily see that the function f given by (26) satisfies $f \in C^1(X;\mathbb{C})$ and $f(0) = Df(0) = 0$. For any w with small $|w|$, we get

$$\begin{aligned} f(\Phi_c w) &= \varepsilon\left(\int_{-\infty}^{0} Q(-\theta)(\Phi_c w)(\theta)d\theta\right)^m + g(\Phi_c w) \\ &= \varepsilon\left(w\int_{-\infty}^{0} Q(-\theta)d\theta\right)^m + o(w^m) = \varepsilon(c_0 w)^m + o(w^m); \end{aligned}$$

hence,

$$\begin{aligned} rH(w) &= f(\Phi_c w + F_*(\phi_c w)) \\ &= f(\Phi_c w) + \{f(\Phi_c w + F_*(\phi_c w)) - f(\Phi_c w)\} \\ &= f(\Phi_c w) + \varepsilon\,\{[L_1(\Phi_c w + F_*(\phi_c w))]^m - [L_1(\Phi_c w)]^m\} + o(w^m) \\ &= \varepsilon(c_0 w)^m + o(w^m) + \varepsilon\sum_{k=0}^{m-1}\binom{m}{k}\{L_1(\Phi_c w)\}^k\{L_1(F_*(\Phi_c w))\}^{m-k}, \end{aligned}$$

here L_1 is a bounded linear functional on L^1_ρ defined by $L_1(\phi) := \int_{-\infty}^0 Q(-\theta)\phi(\theta)d\theta$. Recall that $L_1(F_*(\Phi_c w)) = o(w)$ as $w \to 0$; hence

$$\sum_{k=0}^{m-1}\binom{m}{k}\{L_1(\Phi_c w)\}^k\{F_*(\Phi_c w)\}^{m-k} = o(w^m) \quad \text{as } w \to 0.$$

Thus $rH(w) = \varepsilon(c_0 w)^m + o(w^m)$ as $w \to 0$. Hence it follows that

$$H(w) = (\varepsilon/r)c_0^m w^m + o(w^m) \quad \text{as } w \to 0.$$

Consequently, one can easily see that the zero solution of the central equation of Eq. (25) is uniformly asymptotically stable if $\varepsilon < 0$ and if m is an odd natural number; while it is unstable if $\varepsilon > 0$ and if m is an odd natural number, or if $\varepsilon \neq 0$ and if m is an even natural number. Therefore, by virtue of Theorem 3, we get the following result:

Proposition 9. *Assume that*

$$f(\phi) = \varepsilon\left(\int_{-\infty}^{0} Q(-\theta)\phi(\theta)d\theta\right)^m + g(\phi), \quad \forall\phi \in X, \tag{27}$$

here ε is a nonzero constant, m is a natural number such that $m \geq 2$, Q is a function satisfying $\|Q\|_{1,\rho} < \infty$, $\|Q\|_{\infty,\rho} < \infty$ and $\int_0^\infty Q(t)dt > 0$ and $g(\phi) = o(\|\phi\|_X^m)$ as $\|\phi\|_X \to 0$ with $g \in C^1(X;\mathbb{C})$. Then the following statements hold true;

(i) *if m is odd and $\varepsilon < 0$, then the zero solution of Eq. (25) is uniformly asymptotically stable (in L^1_ρ);*

(ii) *if m is odd and $\varepsilon > 0$, then the zero solution of Eq. (25) is unstable (in L^1_ρ);*

(iii) *if m is even and $\varepsilon \neq 0$, then the zero solution of Eq. (25) is unstable (in L^1_ρ).*

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